End of the inflation time

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Abstract

In this text, we try to show step by step how one can get a never-ending inflationary scenario and thus analyze it to deduce e.g. some regime of theory's parameters. In particular, we consider the Hilltop and Starobinski Inflation. Numerical analysis shows that the Starobinski Inflation does not violate the No Eternal Inflation Principle and $p = 3$ Hilltop model parameters are restricted.

1 Introduction[1]

1.1 General Relativity

General Relativity is a theory of Lorentzian manifolds namely pairs of smooth Hausdorff manifolds and metrics $g_{\mu\nu}$ with Lorentzian signature (1,3), which are smooth with respect to the points of a given manifold. Such models are equivalent up to a diffeomorphism.
We will use following definitions:

$$\begin{aligned}\Gamma^{\mu}_{\nu\xi} =& \frac{1}{2} g^{\mu\eta} \left(\partial_\xi g_{\nu\eta} + \partial_\nu g_{\xi\eta} - \partial_\eta g_{\nu\xi}\right), \\ R_{\mu\nu} =& \partial_\xi \Gamma^{\xi}_{\mu\nu} - \partial_\mu \Gamma^{\xi}_{\xi\nu} + \Gamma^{\xi}_{\xi\eta}\Gamma^{\eta}_{\mu\nu} - \Gamma^{\xi}_{\mu\eta}\Gamma^{\eta}_{\xi\nu}, \\ R =& g^{\mu\nu} R_{\mu\nu}.\end{aligned}$$

Einstein-Hilbert action in a particular spacetime is given by:

$$S_{EH} = \int d^4x \sqrt{|g|} \left(R + \Lambda + \mathcal{L}_m\right),$$

where Λ is cosmological constant (dark energy term, later omitted) and $\mathcal{L}_m$ is a lagrangian density corresponding to matter.

*JR- Sec. 1, JL- Sec. 2, AS- Sec. 3, 5, JC- Sec. 4, 6

[1] The whole chapter is based on this work. (1)

By variation over $g_{\mu\nu}$ of the action we can deduce Einstein equations:

$$R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu},$$

where by definition $\frac{\delta S_m}{\delta g_{\mu\nu}} = \int d^4 x T^{\mu\nu} \delta g_{\mu\nu}$ and S_m is action corresponding only to matter.

1.2 FLRW cosmology

Cosmological models try to capture the whole Universe and its evolution within formalism of General Relativity. Observations, including cosmic microwave background radiation, conclude that our Universe is highly homogeneous and isotropic in large scales. Homogeneity means that the Universe has even distribution of matter and isotropy says that Universe "looks" identically in each direction. The FLRW metrics are the only one satisfying both homogenity and isotropy conditions. They can be written in spherical coordinates as follows:

$$ds^2 = dt^2 - a^2(t) \left(\frac{1}{1 - kR^2} dR^2 + R^2 d\theta^2 + R^2 sin^2\theta d\phi^2 \right),$$

where $k \in \{-1, 0, 1\}$ is describing curvature of spatial dimensions.
Alternatively, we can rewrite it using $d\chi^2 = \frac{1}{1-kR^2} dR^2$ into:

$$\begin{aligned}
ds^2 &= dt^2 - a^2(t) \left(d\chi^2 + sinh^2\chi d\Omega^2 \right), \text{ for } k = -1, \\
ds^2 &= dt^2 - a^2(t) \left(d\chi^2 + \chi^2 d\Omega^2 \right), \text{ for } k = 0, \\
ds^2 &= dt^2 - a^2(t) \left(d\chi^2 + sin^2\chi d\Omega^2 \right), \text{ for } k = 1,
\end{aligned}$$

where $d\Omega^2$ is canonical metric form on $\mathcal{S}^2$. In our Universe curvature is almost negligible and thus we consider $k = 0$.
The energy-momentum tensor of the perfect fluid may be written as:

$$T^\mu_\nu = (\epsilon + p) u^\mu u_\nu - p \delta^\mu_\nu,$$

where ϵ - energy density, p - pressure, u_μ - 4-velocity.
Non-zero Christoffel symbols (of second kind) for FLRW metrics are:

$$\begin{aligned}
\Gamma^i_{0j} =& \frac{1}{2} g^{ik} \frac{\partial g_{jk}}{\partial t} = \frac{\dot{a}}{a} \delta^i_j, \\
\Gamma^0_{ij} =& a\dot{a} g_{ij}.
\end{aligned}$$

Consequently, Ricci tensor has the following nontrival terms:

$$\begin{aligned}
R_{00} =& -3 \frac{\dot{a}}{a}, \\
R_{ij} =& (\ddot{a}a + 2\dot{a}^2 + 2k) h_{ij}
\end{aligned}$$

and thus Ricci scalar is equal to:

$$R = g^{\mu\nu} R_{\mu\nu} = R_{00} - \frac{1}{a^2}\delta^{ij} R_{ij} = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right).$$

Einstein equations can be reformulated as follows:

$$R^\mu_\nu = \kappa\left(T^\mu_\nu - \frac{1}{2}T\delta^\mu_\nu\right),$$

with $T = \epsilon - 3p$, $\kappa = \frac{8\pi G}{c^4}$ and summed up to the Friedmann equations:

$$-3\frac{\ddot{a}}{a} = \frac{\kappa}{2}(\epsilon + 3p),$$
$$\frac{2\dot{a}^2 + a\ddot{a} + 2k}{a^2} = \frac{\kappa}{2}(p - \epsilon).$$

Introducing Hubble parameter $H = \frac{\dot{a}}{a}$, conservation of energy $\nabla^\mu T_{\mu\nu} = 0 \Rightarrow \frac{d\epsilon}{dt} = -3H(\epsilon + p)$ and assuming constant depedence $p = w\epsilon$ we can solve equations above:

$$\epsilon \propto a^{-3(1+w)},$$
$$a(t) \propto e^{Ht}, \text{ for } w = -1,$$
$$a(t) \propto t^{\frac{2}{3(1+w)}} \text{ otherwise.}$$

2 Inflation[2]

In general terms inflation is a stage of accelerated expansion of the Universe during which gravity acts as repulsive force. The end of inflation, called the graceful exit, is marked by the transition from accelerated to decelerated Friedman expansion. Recall the Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varepsilon + 3p).$$

The stage of accelerated expansion can take place when $\ddot{a} > 0$. That necessarily results in violation of strong energy condition $(\varepsilon + 3p) > 0$. The example of such violation is the de Sitter Universe solution. However it does not provide the conditions for the graceful exit. The graceful exit can be accomplished by allowing Hubble parameter varying in time. From the left hand side of the considered equation:

$$\frac{\ddot{a}}{a} = H^2 + \dot{H}$$

one can see the graceful exit happens where $\ddot{a} = 0$, which takes place for $H^2 \sim \dot{H}$. CMB observational data gives constraint on the ratio $\frac{\dot{a}_i}{\dot{a}_0} < 10^{-5}$. Additionally

[2]The chapter is based on (1),(2).

$\frac{\dot{a}_f}{\dot{a}_0} > 10^{28}$, hence:

$$\frac{\dot{a}_i}{\dot{a}_f}\frac{\dot{a}_f}{\dot{a}_0} = \frac{a_i}{a_f}\frac{H_i}{H_f}\frac{\dot{a}_f}{\dot{a}_0} < 10^{-5}$$

and:

$$\frac{a_f}{a_i} > 10^{33}\frac{H_i}{H_f}.$$

Therefore, the ratio of final and initial scale factors is:

$$\frac{a_f}{a_i} \sim \exp\left(H_i t_f\right) > 10^{33}.$$

The duration of inflation can be measured by a number of e-folds. E-fold corresponds to a period in which the Universe expands by a factor e. From the estimates above $t_f > 75 H_i^{-1}$ and inflation lasts roughly more than 75 e-folds.

2.1 Slow roll conditions

In order to formulate the model of inflation followed by the graceful exit one introduces the scalar field called the inflaton. In the classical description of inflation we consider classical scalar field with homogeneous distribution. In that case the action is of the form:

$$S = \int_\Omega d^4x\sqrt{|g|}\left(\frac{1}{2}\dot{\phi}^2 - V\left(\phi\right)\right)$$

and the Klein-Gordon equation in FLRW background is:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0.$$

We can define energy density and pressure:

$$\varepsilon = \frac{1}{2}\dot{\phi}^2 + V\left(\phi\right),$$
$$p = \frac{1}{2}\dot{\phi}^2 - V\left(\phi\right).$$

The equation of state is given by:

$$w = \frac{p}{\varepsilon} = \frac{\frac{1}{2}\dot{\phi}^2 - V\left(\phi\right)}{\frac{1}{2}\dot{\phi}^2 + V\left(\phi\right)}.$$

If $\dot{\phi}^2 \ll V\left(\phi\right)$ then the equation of state takes the form $w \simeq -1 < -\frac{1}{3}$ satisfying condition for accelerated expansion. This is the first slow-roll condition:

$$\dot{\phi}^2 \ll V\left(\phi\right).$$

Klein-Gordon equation has an attractor solution of the form: $\dot{\phi} = V_{,\phi}/3H$ for large friction term. As this is expected for inflation, we get the second slow-roll condition:

$$\ddot{\phi} \ll 3H\dot{\phi}.$$

It might be convenient to introduce two slow-roll parameters ϵ, η:

$$\epsilon = -\frac{\dot{H}}{H^2} \simeq \frac{1}{2}\left(\frac{V_{,\phi}}{V}\right)^2,$$
$$\eta = \frac{\ddot{\phi}}{H\dot{\phi}} \ll 1.$$

Then the slow-roll conditions can be expressed in terms of potential:

$$\epsilon \simeq \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2,$$
$$\eta \simeq M_P^2 \frac{V_{,\phi\phi}}{V} \ll 1.$$

The end of the process of inflation is marked by $\epsilon \simeq 1$. Time of inflation can be derived from the definition:

$$dN = Hdt,$$

where the number of e-folds N is calculated by integration. After taking into consideration the slow-roll conditions, the expression for N takes the form:

$$N = \int_{t_i}^{t_{end}} Hdt = \int_{\phi_i}^{\phi_{end}} \frac{H}{\dot{\phi}} d\phi = \int_{\phi_{end}}^{\phi_i} \frac{V(\phi)}{V_{,\phi}(\phi)} d\phi.$$

From that result we can derive the solution for scale factor:

$$a(\phi) \simeq a_i \exp\left(8\pi \int_{\phi}^{\phi_i} \frac{V}{V_{,\phi}} d\phi\right).$$

2.2 Primordial inhomogeneities

CMB observational data show the existence of inhomogeneities in the Universe of the order $\delta T/T \simeq 10^{-5}$. Inflation preserves homogeneity given the homogeneous state at the start of the expansion. However, it can generate inhomogeneities from quantum fluctuations. In cosmological models preceding inflation inhomogeneities were postulated and the initial conditions were set to match the observational data. The advantage of the inflationary cosmology is the ability to explain the origin of the inhomogeneities and calculate their spectrum. It seems convenient to describe inhomogeneities as random fields with each Fourier component having random Gaussian distribution with variance $\sigma_k^2 \equiv |\Phi_k|^2$. The expression for power spectrum is of the form:

$$\mathcal{P}_S = \frac{|\Phi_k|^2 k^3}{2\pi^2}.$$

We can define spectral index as:

$$n_s - 1 = \frac{d \ln \mathcal{P}_S}{d \ln k}.$$

2.2.1 Scalar perturbations

Consider the general form of action for scalar field:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{|g|} p\left(X, \phi\right)$$

with:

$$X = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

and the FLRW metric with $k = 0$. The equations for background parameters ϕ_0 and $a\left(t\right)$ are:

$$\begin{aligned} H^2 &= \frac{8\pi G}{3} \varepsilon, \\ \dot{\varepsilon} &= -3H\left(\varepsilon + p\right). \end{aligned}$$

The ratio of partial derivatives of pressure and energy density is called speed of sound for perturbations:

$$c_s^2 = \frac{p_{,X}}{\varepsilon_{,X}} = \frac{\varepsilon + p}{2X\varepsilon_{,X}}.$$

In the case of the canonical scalar field, the speed of sound is equal to the speed of light $c_S = 1$. We can describe the inhomogeneities in the field as follows:

$$\phi\left(t, x\right) = \phi_0\left(t\right) + \delta\phi\left(t, x\right).$$

The field inhomogeneities induce metric perturbations:

$$ds^2 = dt^2\left(1 + 2\Phi\right) - a\left(t\right)^2 \delta_{ij}\left(1 - 2\Phi\right) dx^i dx^k.$$

We can introduce new variables ζ, ξ defined by relations:

$$\begin{aligned} \Phi a &= 4\pi G H \xi, \\ \frac{\delta\phi}{\dot{\phi}} &= \frac{\zeta}{H} - \frac{4\pi G}{a} \xi. \end{aligned}$$

It is possible to write linear equations for ζ and ξ:

$$\begin{aligned} \dot{\xi} &= \frac{a\left(\varepsilon + p\right)}{H^2} \zeta, \\ \dot{\zeta} &= \frac{c_s^2 H^2}{a^3\left(\varepsilon + p\right)} \Delta\xi. \end{aligned}$$

With the definition of another variables:

$$\begin{aligned} z &= \frac{a\left(\varepsilon + p\right)^{\frac{1}{2}}}{c_s H}, \\ v &= z\zeta \end{aligned}$$

one can see v is a canonical quantisation variable with Klein-Gordon equation:

$$v'' - c_s^2 \Delta v - \frac{z''}{z} v = 0.$$

ζ is related to metric perturbations Φ by:

$$\zeta = \frac{5\varepsilon + 3p}{3\varepsilon + p} \Phi + \frac{2}{3} \frac{\varepsilon}{\varepsilon + p} \frac{\dot{\Phi}}{H}$$

and its power spectrum is of the form:

$$\mathcal{P}_\zeta = \frac{1}{2\pi^2} |\zeta_k|^2 k^3 = \frac{k^3}{2\pi^2} \frac{|v_k|^2}{z^2}.$$

2.2.2 Tensorial perturbations

The action for traceless tensor perturbations h^i_j is expressed by:

$$S = \frac{1}{64\pi G} \int a^2 \left(h^{i'}_j h^{j'}_i - h^i_{j,l} h^{j,l}_i \right) d\eta d^3 x.$$

Fourier expansion of the perturbation has the form:

$$h^i_j (\boldsymbol{x}, \eta) = \int h_{\boldsymbol{k}} (\eta) \, e^i_j (\boldsymbol{k}) \, e^{i\boldsymbol{k}\boldsymbol{x}} \frac{d^3 k}{2\pi^{3/2}}.$$

If we define variable v_k:

$$v_k = \sqrt{\frac{e^i_j e^j_i}{32\pi G}} a h_{\boldsymbol{k}},$$

then the action for perturbations can be reformulated:

$$S = \frac{1}{2} \int \left(v'_{\boldsymbol{k}} v'_{-\boldsymbol{k}} - \left(k^2 - \frac{a''}{a} \right) v_{\boldsymbol{k}} v_{-\boldsymbol{k}} \right) d\eta d^3 x$$

with equation of motion for $v_{\boldsymbol{k}}$:

$$v''_{\boldsymbol{k}} + \omega_k^2 (\eta) \, v_{\boldsymbol{k}} = 0,$$
$$\omega_k^2 (\eta) = k^2 - \frac{a''}{a}.$$

The expression for power spectrum of tensorial perturbations is:

$$\mathcal{P}_h = \frac{16 |v_{\boldsymbol{k}}|^2 k^3}{\pi a^2}.$$

If we use the approximation that gravitational waves depend weakly on equation of state and set $H = H_\Lambda$ then we obtain the solution for $v_{\boldsymbol{k}}$:

$$v_{\boldsymbol{k}} (\eta) = \frac{1}{\sqrt{k}} \left(1 + \frac{i}{k\eta} \right) \exp \left(ik \left(\eta - \eta_i \right) \right)$$

and the power spectrum:

$$\mathcal{P}_h = \frac{8H_\Lambda^2}{\pi}[1 + (k\eta)^2].$$

In the long wavelength case: $c_s k \simeq Ha$, the power spectrum takes the form:

$$\mathcal{P}_h \simeq \frac{16H^2}{\pi}$$

and the spectral index is:

$$n_T = \frac{d\ln \mathcal{P}_h}{d\ln k} \simeq -3\left(1 + \frac{p}{\varepsilon}\right).$$

Tensor-to-scalar ratio is defined as:

$$r = \frac{\mathcal{P}_h}{\mathcal{P}_S} \simeq 24[c_S\left(1 + \frac{p}{\varepsilon}\right)].$$

3 Starobinsky Inflation

In 1980 Starobisky (3) proposed a model where a pure modified gravitational action can cause non-singular evolution of the Universe, namely:

$$S = \frac{1}{2}\int \sqrt{|g|}d^4x\left(M_p^2 R + \frac{1}{6M^2}R^2\right), \tag{1}$$

where M is some "mass" parameter, with value taken to fit the Planck data. Now we will rewrite the action into equivalent linear representation:

$$S_l = \frac{1}{2} = \frac{1}{2}\int \sqrt{|g|}d^4x\left(\frac{M_p^2}{2}R + \frac{1}{M}R\psi - 3\psi^2\right), \tag{2}$$

if we write equations of motion for ψ we obtain:

$$\frac{1}{M}R = 6\psi. \tag{3}$$

Then if we use a following conformal transformation:

$$g_{\mu\nu} \to e^{-\sqrt{2/3}\phi/M_p}g_{\mu\nu} = \left(1 + \frac{2\psi}{MM_p^2}\right)g_{\mu\nu} \tag{4}$$

we get action with scalar field coupled to gravity:

$$S = \frac{1}{2}\int d^4x\sqrt{|g|}\left(\frac{M_p^2}{2}R + \frac{1}{2}\partial_\mu\partial^\mu\phi - \frac{3}{4}M_p^4 M^2\left(1 - e^{-\sqrt{2/3}\phi/M_p}\right)\right). \tag{5}$$

R^2 Term gives equivalent solutions as the evolution of scalar field with exponential type potential. According to Planck data, Starobinsky model and its descendants are the main class of models which has correct tensor to scalar ratio and scalar-tilt:

$$n_s - 1 \approx -\frac{2}{N} \qquad\qquad r \approx \frac{12}{N^2}, \tag{6}$$

with N being the number of e-folds.

4 Eternal inflation

In this section, based on (4), we will show, under what circumstances the inflation does not end. This behavior is called eternal inflation and provides restrictions on free parameters in popular inflationary models.

Consider a classical solution to Einstein Equations in inflationary FLRW metric, in the slow-roll approximation:

$$3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \qquad\qquad H^2 M_{Pl}^2 = \frac{1}{3} V(\phi).$$

The background field method provides a description of the fluctuating quantum field in the classical background:

$$\phi(t, \vec{x}) = \phi_{cl}(t, \vec{x}) + \delta\phi(t, \vec{x}).$$

Fluctuation are normally distributed and averaged over the Hubble volume. Equation of motion for the full field takes form of slow roll equation, with additional noise term:

$$3H\dot{\phi} + \frac{\partial V}{\partial \phi} = N(t),$$

where $N(t)$ is a Gaussian distribution with mean equal 0 and variance $\sigma = \frac{H^3 t}{4\pi^2}$.

4.1 Field evolution

In the covered cases, the inflaton decays exponentially with time. To see this, consider a set of scalar fields: $\phi = \{\phi_1, \dots, \phi_n\}$. Time evolution of the fields' probability density is given by the Fokker-Planck equation:

$$\dot{P}[\phi, t] = \frac{1}{2}(\frac{H^3}{4\pi^2})\partial_i \partial^i P[\phi, t] + \frac{1}{3H}\partial_i \left(\partial^i V(\phi) P[\phi, t]\right), \tag{7}$$

where $\partial_i := \frac{\partial}{\partial \phi^i}$, $\dot{P}[\phi, t] := \frac{\partial}{\partial t} P[\phi, t]$. Above equation is difficult to solve for a general potential $V(\phi)$, but it can be solved for a sum-separable potential:

$$V(\phi) = V_0 + \sum_{i=1}^{N} V_i\left(\phi^i\right), \tag{8}$$

with $V_i\left(\phi^i\right)$ linear or quadratic, under the assumption that H is constant, i.e.

$$H^2 M_{Pl}^2 = \frac{V_0}{3}. \tag{9}$$

Under these assumptions, the solution takes a simple multivariate Gaussian form:

$$P[\phi, t] = \prod_{i=1}^{N} P_i[\phi_i, t], \tag{10}$$

with

$$P_i[\phi_i, t] = \frac{1}{\sigma_i\left(t\right)\sqrt{2\pi}} \exp\left[-\frac{\left(\phi_i - \mu_i\left(t\right)\right)^2}{2\sigma_i\left(t\right)^2}\right]. \tag{11}$$

Case 1: Free Massless Field $V\left(\phi\right) = V_0$
For a constant potential equation (7) is solved by a Gaussian distribution (11) with

$$\mu\left(t\right) = 0, \qquad \sigma^2\left(t\right) = \frac{H^3}{4\pi^2}t. \tag{12}$$

A delta-function distribution initially centered at $\phi = 0$ will remain centered at $\phi = 0$ for all time, however, it will spread out by an amount $\sigma\left(t = H^{-1}\right) = H/2\pi$ after a Hubble time. This represents the standard "Hubble-sized" quantum fluctuations that are well-known in the context of inflation, famously imprinting in the CMB and ultimately seeding the observed large-scale structure.
Case 2: Linear Potential $V\left(\phi\right) = V_0 - \alpha\phi$
For the linear potential Fokker-Planck is again solved by a the Gaussian distribution (11) with:

$$\mu\left(t\right) = \frac{\alpha}{3H}t, \qquad \sigma^2\left(t\right) = \frac{H^3}{4\pi^2}t, \tag{13}$$

the time-dependence of $\mu\left(t\right)$ is explained by the classical rolling of the field in the linear potential, which is governed by the slow-roll equation of motion

$$3H\dot{\phi} = -\frac{\partial V}{\partial \phi} = \alpha. \tag{14}$$

The time-dependence of $\sigma^2\left(t\right)$ is due purely to Hubble-sized quantum fluctuations, and indeed it precisely matches the result in the free massless case.
For a linear and quadratic potential the equation simplifies to the heat equation, hence the solutions are gaussian. Near the top of the hill, the potential can be approximated by a constant, then the solving distribution is of the form:

$$P[\phi, t] = \sqrt{\frac{2\pi}{H^3 t}} \exp\left(-\frac{2\pi^2\phi^2}{H^3 t}\right),$$

assuming initially it was delta-like centered at $\phi = 0$. The probability distribution is decreasing with time, it remains centered at the initial position. For linearly decreasing potential $V(\phi) = V_0 - \alpha\phi$ the solution is:

$$P[\phi, t] = \sqrt{\frac{2\pi}{H^3 t}} \exp\left(-\frac{2\pi^2 \left(\phi - \frac{\alpha}{H} t\right)^2}{H^3 t}\right),$$

notice the mean value of the field increases linearly with time. This fact is understood as field rolling down the hill's slope according to the slow-roll equation: $3H\dot{\phi} = \alpha$.

4.2 Eternal inflation Conditions

Under what conditions the eternal inflation occurs? To answer this question, consider once again linear potential. The condition for the slow-roll is:

$$\epsilon_V := \frac{M_{Pl}^2}{2} \left(\frac{\alpha}{V(\phi)}\right)^2 \ll 1.$$

It is satisfied for $\phi < 0$, assuming $M_{Pl}\alpha \ll V_0$. The field increases with time, due to the equation of motion, hence by the time $\phi_c = \frac{V_0}{\alpha} - \frac{M_{pl}}{\sqrt{2}}$ slow-roll will stop $\epsilon_V \approx 1$. The Probability density tends to 0, when $t \longrightarrow \infty$, thus the probability that inflation still lasts is:

$$Pr[\phi < \phi_c] = \int_{-\infty}^{\phi_c} d\phi P[\phi, t]$$

also vanishes at sufficiently long times. It appears that from the point of the Fokker-Planck equation inflation stops at some point. However, there is an additional effect to be included- expansion of the Universe. Assuming the initial volume of the Universe to be U_0, the size of the Universe depends on time according to:

$$U(t) = U_0 e^{3Ht}.$$

One can interprete the probability $Pr[\phi < \phi_c]$ as fraction of the volume $U(t)$ still inflating, that is:

$$U_{inflating}(t) = U_0 e^{3Ht} Pr[\phi < \phi_c].$$

Evaluating the integral for probability density, in the linear case gives:

$$Pr[\phi < \phi_c] = \frac{1}{2}\text{erfc}\left(\frac{\frac{\alpha}{3H}t - \phi_c}{\frac{H}{2\pi}\sqrt{2Ht}}\right).$$

The error function may be approximated by an exponential:

$$Pr[\phi < \phi_c] = C(t) \exp\left(-\frac{4\pi^2\alpha^2}{18H^5} t\right),$$

where $C(t)$ is power-law in t. ϕ_c vanished from the final approximation of the probability, this occurs also in theories with more complicated potentials. By comparing the exponentials we can check, whether $U_{inflating}$ will grow or tend to zero. The condition for eternal inflation to occur becomes:

$$3H > \frac{4\pi^2\alpha^2}{18H^5},$$

now using slow-roll EOM: $H^2 M_{Pl}^2 = \frac{V}{3}$, and for linear potential $\alpha = V'(\phi)$ above condition can be rewritten:

$$\frac{|V'|}{V^{\frac{3}{2}}} < \frac{\sqrt{2}}{2\pi}\frac{1}{M_{Pl}^2}.$$

This can be interpreted as quantum fluctuations dominating over classical field rolling. For linear potential, this is satisfied for a large ϕ.

4.3 Numerical analysis

The set of analytical solutions to the Fokker-Planck equation is not vast. For the potential given by the hill-top model, $V(\phi) = V_0 - \frac{\alpha}{p}\phi^p$ only for linear and quadratic cases exact solutions can be found. For higher powers of ϕ one needs to rely on numerical solutions. Discretizing the Langevin equation gives:

$$\phi_n = \phi_{n-1} - \frac{1}{3H}V'(\phi_{n-1})\,\delta t + \delta\phi_q(\delta t),$$

with $\delta\phi_q(\delta t)$ being random number taken from the gaussian distribution with mean equal zero, and variance $\frac{H^3}{4\pi^2}\delta t$. The initial condition was chosen to be $\phi_0 = 0$, while the time step is $\delta t = \frac{1}{100H}$. Since the value of ϕ_c is irrelevant one should check in every iteration, if the fluctuating field is bigger than some arbitrary value (in our analysis Ht) and calculate the probability after solving the equation large number of times. For Starobinsky inflation the potential is of form:

$$V(\phi) = V_0\left(1 - \exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}\right)^2\right).$$

In this case, the probability does not decay exponentially, which can be seen on fig. 3 For a constant Hubble parameter[3] 10 values of α has been investigated, and the results have been plotted on Fig. 2. The decay parameter Γ is defined as follows:

$$Pr[\phi < \phi_c] \sim e^{-\Gamma t}.$$

For $p = 3$ the critical value of α_c is 670. The same analysis has been performed in case of H fluctuating due to equations of motion, the similar critical value was found $\alpha_c = 678$.

[3]Even though H can be chosen to be arbitrary constant in the numerical solution, the relation $H^2 M_{Pl}^2 = V(\phi)$ needs to be satisfied

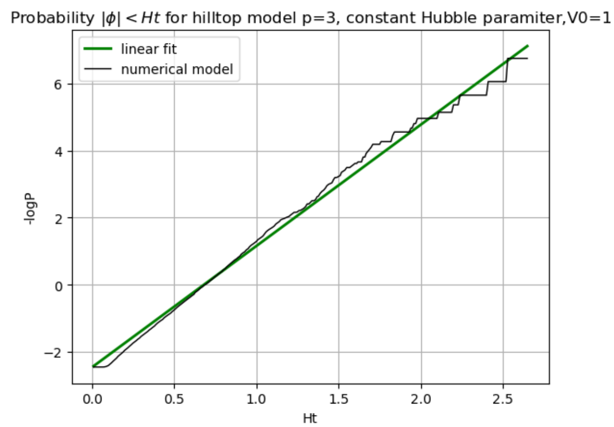

Figure 1: Solution to the Langevin equation in hilltop model with $p = 3$, $\alpha = 1000$. Notice the exponential decay.

From our considerations we see, that for $p = 3$ hilltop models with $\alpha_c < 670$ are excluded because they predict Eternal Inflation. In the Starobinsky Inflation, this is not the case, and inflation ends independently of the parameters in the theory.

5 Analytic solutions to the Fokker-Planck Equation

The study of inflation begins by considering a scalar field theory in a quasi-de Sitter background:

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{15}$$

with

$$ds^2 = -dt^2 + e^{2Ht}dx. \tag{16}$$

The dynamical equations describing the evolution of the scalar field and the background geometry are given by

$$3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \qquad H^2 M_{Pl}^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{17}$$

The action for the quantum fluctuations is quadratic, so the fluctuations will be Gaussian. These fluctuations are averaged over a Hubble volume by defining a

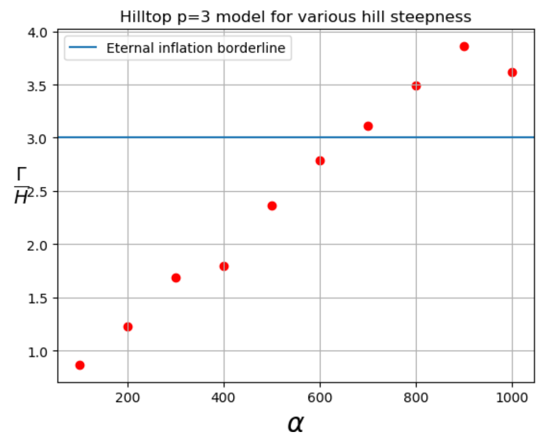

Figure 2: Decay parameter scales linearly with α, at values bigger than $\alpha_c \approx 670$ eternal inflation does not occur

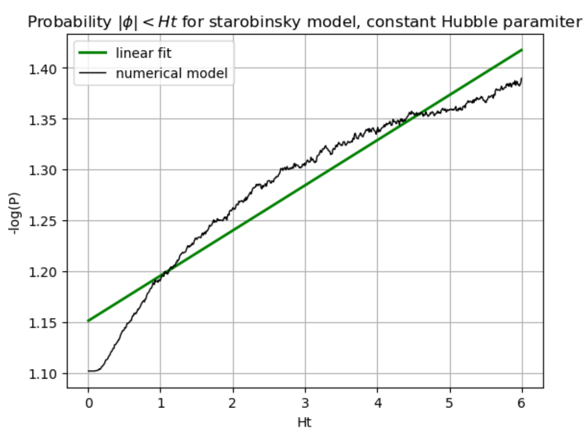

Figure 3: Solution to the Langevin equation, for Starobinsky Inflation.

smeared field

$$\delta\phi_H(t) = \int \frac{d^3k}{(2\pi)^3}\sigma_t(k)\,\delta\phi_k(t)\,, \tag{18}$$

with $\delta\phi_k$ a Fourier mode of $\delta\phi$ and $\sigma_t(k)$ a smearing function that corresponds to averaging over one Hubble volume at a time t. The average size of these Gaussian fluctuations is given by

$$< [\delta\phi_H(t) - \delta\phi_H(0)]^2 >= \left(\frac{H}{2\pi}\right)^2 Ht, \tag{19}$$

where

$$\delta\phi_q = \frac{H}{2\pi}. \tag{20}$$

Case 3: : Free Massive Field $V(\phi) = V_0 + \frac{1}{2}m^2\phi^2$
Equation (7) is solved by a Gaussian distribution (11t) with

$$\mu(t) = 0, \qquad \sigma^2(t) = \frac{3H^4}{8\pi^2 m^2}\left(1 - \exp\left[-\frac{2m^2}{3H}t\right]\right) \tag{21}$$

The central value of the Gaussian distribution remains fixed at $\phi = 0$ for all time. In the limit $m/H \to 0$, we may expand the exponential to linear order to recover the formula for $\sigma^2(t)$ for the free massless field case in (12).
Case 4: Tachyonic Field $V(\phi) = V_0 - \frac{1}{2}m^2\phi^2$ The equation (7) is solved by Gaussian distribution(11) by:

$$\mu(t) = 0, \qquad \sigma^2(t) = \frac{3H^4}{8\pi^2 m^2}\left(-1 + \exp\left[\frac{2m^2}{3H}t\right]\right) \tag{22}$$

The central value of the Gaussian distribution stays at $\phi = 0$.

6 Conclusion

In the work, we have presented various models of inflation with appropriate theoretical introduction. The Principle of No Eternal Inflation provided a valuable tool for excluding theories which do not predict the end of inflation. Detailed numerical analysis of the hilltop $p = 3$ model has been performed and the results obtained in (4) has been verified. The lower boundary of the proportionality parameter α has been estimated $\alpha_c = 678$. The discussed Starobinsky Inflation does not predict Eternal Inflation and remains a valid candidate for the history of the Universe.
Program created for the numerical analysis works for any given potential term and may be used to predict Eternal Inflation in future work on more complex theories.

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